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AUTOCORRELATION AND POWER SPECTRUM ANALYSIS.(U)

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AUTOCORRELATION AND POWER SPECTRUM ANALYSIS

by
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USL Problem No. 1-055-00-00

USL Technical Memorandum No. 913-52-65

1 April 1965

Technical memo

USL-TM-913-52-65

14 p.

INTRODUCTION

A Fortran program designated as USL Program No. 0293 has been written by the Digital Computing Branch for the purpose of carrying out a generalized harmonic analysis of a digitized time function. This analysis procedure, as well as the program itself, will be described in this memorandum.

ANALYSIS

A Fourier transform pair relationship exists between the autocovariance function and the power spectral density of a stationary random process x(t). Using this association a method for obtaining an estimate of the power spectrum of the process by transforming the autocovariance function has been developed by Blackman and Tukey.[1] Basically this analysis consists of first calculating the normalized autocovariance function

$$r(T) = \overline{x'(t)\, x'(t+T)} \qquad (1)$$

of the sample function x'(t) and modifying this function using the Hamming lag window[2]. x'(t) is only one member of the ensemble of functions that make up the random process x(t). Consequently it is only possible to obtain estimates of the power spectral density of the random process by performing a Fourier transformation of this modified correlation function.

1 R. B. Blackman and J. W. Tukey, "The Measurement of Power Spectra from the Point of View of Communications Engineering", 1958

2 Other such windows are discussed but the Hamming window is used in the program described by this memorandum.

Since x'(t) is only available for a finite time T, r(T) can only be calculated for /T/ = 0, ..., Tm, where Tm is the maximum lag time, and is not defined outside this region. The Fourier transform of r(T) is given by

$$S(f) = \int_{-\infty}^{\infty} r(T)\, e^{-j\omega T}\, dT \qquad (2)$$

In order to carry out this integration, r(T) must be defined for all T in $(-\infty, \infty)$. By considering r'(T) as a product of r(T) with a weighting function or lag window W(T) where

$$W(T) = \begin{cases} 1 & |T| \leq T_m \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

r'(T) is defined for all T and its Fourier transform may be calculated. Since multiplication in the time domain corresponds to convolution in the frequency domain, the resulting estimated power spectrum is given by

$$P(f) = S(f) \ast W(f) \qquad (4)$$

where S(f) is defined by eq. (2) and

$$W(f) = \int_{-\infty}^{\infty} W(T)\, e^{-j\omega T}\, dT = 2T_m \frac{\sin \omega T_m}{\omega T_m} \qquad (5)$$

W(f) has very undesirable minor lobes which contribute errors to the power spectral density estimates. For this reason other weighting functions have been examined and are described in footnote (1). The particular lag window being used in Program No. 0293 is the Hamming window given by

$$h(T) = \begin{cases} .54 + .46 \cos \frac{\pi T}{T_m} & |T| \leq T_m \\ 0 & |T| > T_m \end{cases} \qquad (6)$$

A comparison of $h(T)$ and $w(T)$ is shown in Figure 1. The Hamming window was chosen because it has the property that its Fourier transform

$$H(f) = .54\, W(f) + .23W(f + \frac{1}{2Tm}) + .23W(f - \frac{1}{2Tm}) \qquad (7)$$

has minor lobes that are no more than 2 per cent of the height of the major lobe, a vast improvement over $W(f)$ which has side lobes as high as 20 per cent of the major lobe. These differences are illustrated in Figure 2. In addition to this important property the Hamming window is easy to work with computationally.

Once having selected a weighting function, power spectral density estimates are obtained by performing a Fourier transformation on the product $r(T) \cdot h(T)$. Since both functions are even, a cosine transformation is all that is required.

In practice, and in this program, continuous data is not available but instead discrete values of the sample function $x'(t)$ at equal increments Δt are available for analysis. In this situation, calculations are simplified by the following approach. First form mean lagged products $r(Tp)$ $p = 0, 1, \ldots m$, (the discrete analog of $r(T)$) where $Tm = m\Delta t$. Next perform a finite cosine series transform using these $m + 1$ correlation coefficients giving raw estimates of the power spectral density $EL(p)$. These estimates are taken equally spaced at $p = 1/2Tm = 1/2m\Delta t$. These raw estimates correspond to using the rectangular weighting function $w(T)$ described in the continuous case. Just as a modification was required in the continuous case to reduce the errors imposed by the large minor lobes of $W(f)$, so too smoothed estimates of the power spectral density must be calculated in the discrete case. The only major difference being that in the previous case the modification was carried out before transforming in order to avoid convolution in the frequency domain. With discrete samples, however, computations are easily carried out in the frequency domain. Since the two functions being convolved are just impulses at equal spacings convolution results in another set of impulses. More specifically for each raw estimate $EL(p)$ convolution results in smoothed estimates Sp where

$$S(p) = .23\, EL(p-1) + .54\, EL(p) + .23\, EL(p+1) \qquad (8)$$

RESOLUTION AND STABILITY

Using the method outlined, estimates of the power spectral density are obtained every $\frac{1}{2Tm}$ cps. These estimates are considered to be an average over frequency of the power in the band 1/2Tm cps. As seen in equation 8, and in Figure 2, adjacent estimates are not independent so the resolution is taken as $2\left(\frac{1}{2Tm}\right) = \frac{1}{Tm}$ cps.

The stability of these estimates is related to the number of degrees of freedom associated with each estimate. If the true spectrum is relatively smooth, degrees of freedom may be approximated by

$$K \cong \frac{2T'}{Tm} \tag{9}$$

where $T' = T - 1/3\ Tm$.

Call the estimates of the power spectral density S^2 and let σ^2 be the true power spectral density. The random variable KS^2/σ^2 has a X^2 distribution with K degrees of freedom. Using the table of X^2 values, it is possible to calculate confidence limits for σ^2 the true power spectral density. The 90% confidence limits are given by

$$\frac{KS^2}{X_2^2} \leq \sigma^2 \leq \frac{KS^2}{X_1^2} \quad \text{where} \tag{10}$$

X_1^2 and X_2^2 are such that $P(X^2 \geq X_1^2) = .95$ and $P(X^2 \geq X_2^2) = .05$ For example, if $K = 25$ degrees of freedom, then 90% of the time the true estimate σ^2 will fall in the interval given by $.66S^2 \leq \sigma^2 \leq 1.71S^2$

Since the length of the sample function x'(t) is limited to T seconds and stability is dependent upon keeping the ratio T/Tm large, a compromise is always necessary between stability and resolution. As Tm increases, so does the resolution, but the stability or confidence in the estimates of the power spectrum decreases. For a fixed T, a value of Tm will be determined for each analysis to give the required resolution with the accuracy needed, however, a rule of thumb is that Tm should never be greater than 5 or 10 per cent of T thus always allowing at least 20 degrees of freedom for each estimate.

Suppose it is desired to analyze frequencies from 0 to some f_{max} cps. If the true spectrum is zero outside f_{max} then a sampling rate of $\Delta t = \frac{1}{2 f_{max}}$ is sufficient. However, if the data is filtered and the frequency response of the filter drops off gradually outside f_{max} then perhaps a sampling rate of $\Delta t = \frac{1}{3 f_{max}}$ should be used to guard against the effect of aliasing of frequencies above f_{max}. Suppose a resolution of 1 cps is desired and additionally the accuracy of 50 degrees of freedom is sufficient, then

$$K = \frac{2T'}{Tm} = 50 \quad \text{and} \quad \frac{1}{Tm} = 1 \text{ cps} \tag{11}$$

Therefore Tm = 1 second and T' = 25 seconds and $T \cong 25$ seconds. This means that if, for example, f_{max} = 100 cps and $\Delta t = \frac{1}{300}$, then 7500 samples would be required. A summary of these calculations is shown in Figure 3 for both the general case and a specific example.

DESCRIPTION OF THE PROGRAM

The program is written to accept both gapless (Datrac) and binary coded decimal (BCD) tape input. Sense switch 2 governs the mode of the input. A maximum sample of 10,000 words can be handled. A variable format for BCD data permits the input to be in any form that is specified on the input format card. The mean and standard deviation are first calculated according to the following standard equations:

$$\text{mean} = \text{XBAR 1} = \frac{1}{N} \sum_{i=1}^{N} X_i \tag{12}$$

$$\text{Standard deviation} = \text{SIG X 1} = \left[\frac{1}{N}\left(\sum_{i=1}^{N} X_i^2 - N\bar{X}^2\right)\right]^{1/2} \tag{13}$$

and printed out. It should be noted that the program does not normalize the data to have a zero mean. This cannot be done because of storage limitations and must be done prior to input to this program. USL Program No. 0444 may be used to normalize BCD data.

Correlation coefficients as a function of lag are computed

according to one of the following formulas as determined by sense switch 4:

$$R(p) = \frac{(N-p)\sum_{i=1}^{N-p} X_i X_{i+p} - \sum_{i=1}^{N-p} X_{i+p} \sum_{i=1}^{N-p} X_i}{\left\{\left[(N-p)\sum_{i=1}^{N-p} X_i^2 - \left(\sum_{i=1}^{N-p} X_i\right)^2\right]\cdot\left[(N-p)\sum_{i=1}^{N-p} X_{i+p}^2 - \left(\sum_{i=1}^{N-p} X_{i+p}\right)^2\right]\right\}^{1/2}} \tag{14}$$

$$R(p) = \frac{N\sum_{i=1}^{N} X_i X_{i+p} - \sum_{i=1}^{N} X_{i+p} \sum_{i=1}^{N} X_i}{\left\{\left[N\sum_{i=1}^{N} X_i^2 - \left(\sum_{i=1}^{N} X_i\right)^2\right]\cdot\left[N\sum_{i=1}^{N} X_{i+p}^2 - \left(\sum_{i=1}^{N} X_{i+p}\right)^2\right]\right\}^{1/2}} \tag{15}$$

where $p = 0, 1 \ldots m$ and m is the maximum lag. In equation 14 the number of data points used to calculate the correlation coefficients must decrease as L increases. In order to use equation 15, the number of points N, used to calculate R(p), remains fixed but the sample size read into the computer must be greater than or equal to N+L. The difference between these equations should be small because N will be large compared with m. Sense switch 1 will give a printed output of R(p) and also output on a plotter tape.

Raw estimates of the power spectrum EL (p) are calculated by using a subroutine called Forer. These estimates are the result of a Fourier cosine transformation of the function R(p), and are given by

$$EL(p) = 1+2\sum_{q=1}^{m-1} R(q)\cos\frac{qp\pi}{m} + R(m)\cos\pi p \qquad p=0,\ 1,\ \ldots\ m \tag{16}$$

Finally, these raw estimates are smoothed using the Hamming coefficients to obtain the results:

$$S(p) = .23EL(p-1) + .54EL(p) + .23EL(p+1) \tag{17}$$

$$\text{where} \quad S(-1) = S(1) \quad \text{and} \quad S(M+1) = S(M-1)$$

Sense switch 6 makes it possible to obtain an average correlation function for a set of K digitized sample functions $\{X_i(t)\}$ by finding the arithmetic mean of the individual normalized correlation functions. Power spectrum estimates are then found using equations 16 and 17 to

operate on this average function. This procedure results in a decrease in the relative variation of the estimates of the power spectrum of the random process x(t) by a factor of approximately $\sqrt{K}$, as illustrated in Figure 4. The solid curve represents the estimate of the power spectral density of a band of white Gaussian noise as obtained from a single sample function, whereas the dotted curve was obtained by averaging 10 samples in the manner described above.

When using sense switch 6, both the individual correlation functions and the average may be printed out but only the average function and its transform are available on a plotter tape.

A simplified flow chart given in Figures 5 and 6 summarizes the order in which all calculations are performed in the program.

SUMMARY

An outline of a method for estimating the power spectrum of a stationary random process from a digitized sample function as developed by Blackman and Tukey has been given. A Fortran program has been written utilizing this approach. Parameters that must be considered when planning and before implementing such an analysis have been defined and emphasis has been placed upon the limitations imposed on the resolution that can be obtained from this type of analysis.

ACKNOWLEDGMENT

The author would like to thank Messrs. W. R. Roth and D. F. Kass for writing and revising this computer program.

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Mathematician

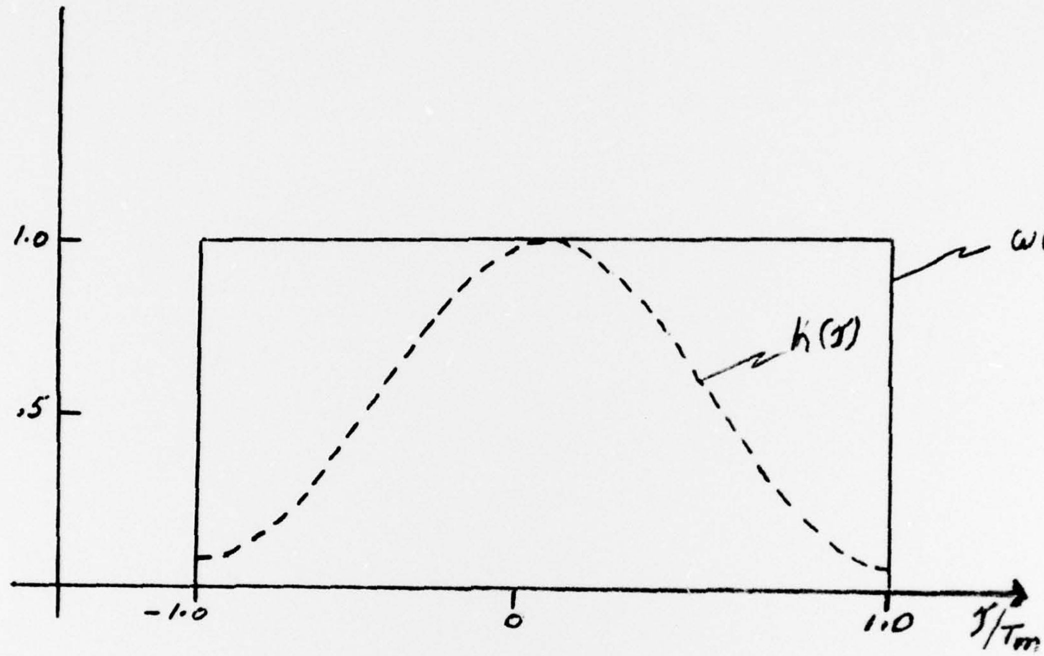

Figure 1

COMPARISON OF THE HAMMING LAG WINDOW

AND

THE UNIFORM WEIGHTING FUNCTION

$$W(f) = 2Tm \frac{\sin 2\pi fTm}{2\pi fTm}$$

$$H(f) = .54W(f) + .23W\left(f + \frac{1}{2Tm}\right) + .23W\left(f - \frac{1}{2Tm}\right)$$

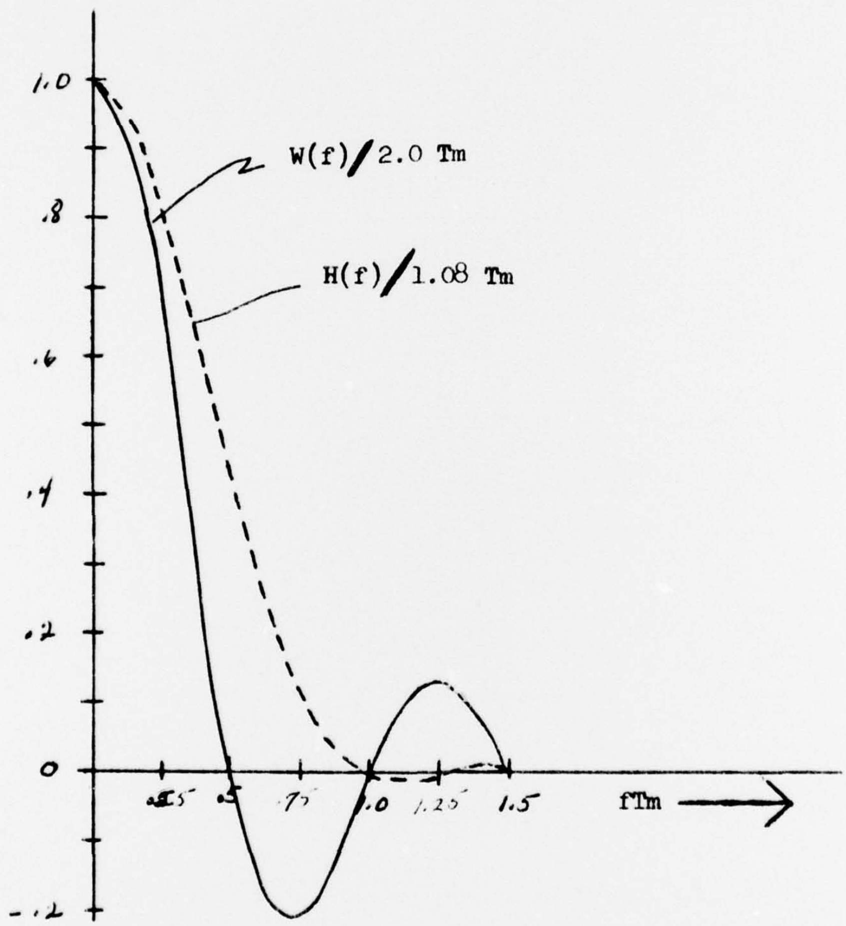

Figure 2

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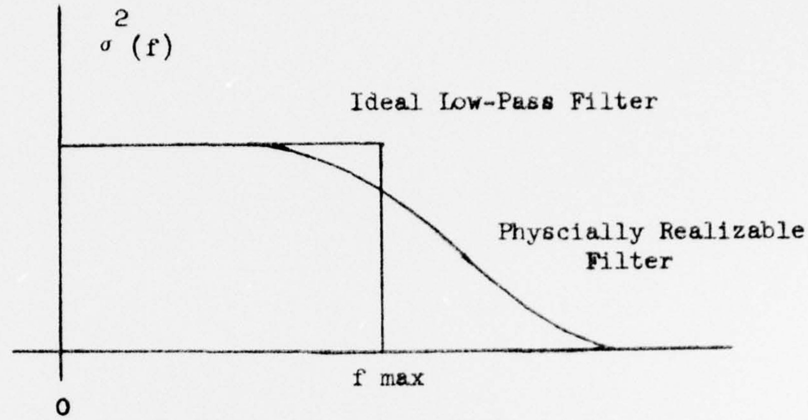


	IDEAL LOW PASS	EXAMPLE f_{max}=50 cps	PHYSICALLY REALIZABLE FILTER	EXAMPLE f_{max}=50 cps
SAMPLING RATE	$\Delta t = \frac{1}{2fmax}$ sec	.01 sec	$\Delta t = \frac{1}{3fmax}$ sec	.0067
SAMPLES / SEC	# = $1/\Delta t$	100/sec	# = $1/\Delta t$	150/sec
RESOLUTION	xcps = 1/Tm	DESIRE .5 cps	xcps = 1/Tm	DESIRE .5 cps
DEGREES OF FREEDOM	$K = \frac{2T'}{Tm}$	DESIRE 30 d.f.	$K = \frac{2T'}{Tm}$	DESIRE 30 d.f.
MAXIMUM LAG	Tm = m.Δt	1/.5 = 2sec	Tm = M.Δt	1/.5 = 2sec
NUMBER OF LAGS	M	200	M	300
APPROXIMATE SAMPLE LENGTH	T'=1/2K.Tm	30 sec	T'=1/2K.Tm	30 sec
APPROXIMATE NUMBER OF SAMPLES	2 fmax T'	30,000	3 fmax T'	45,000

Figure 3

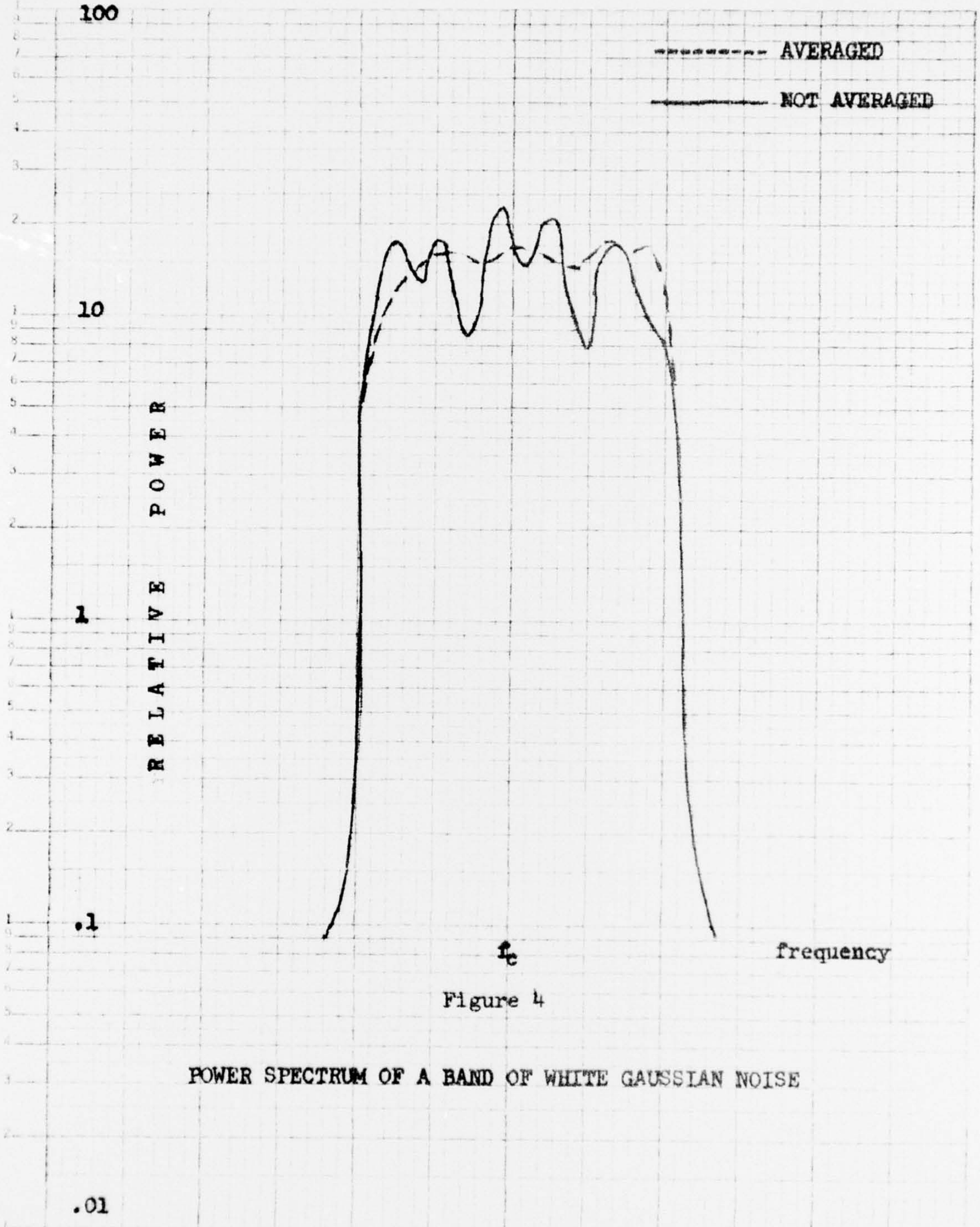


Figure 4

POWER SPECTRUM OF A BAND OF WHITE GAUSSIAN NOISE

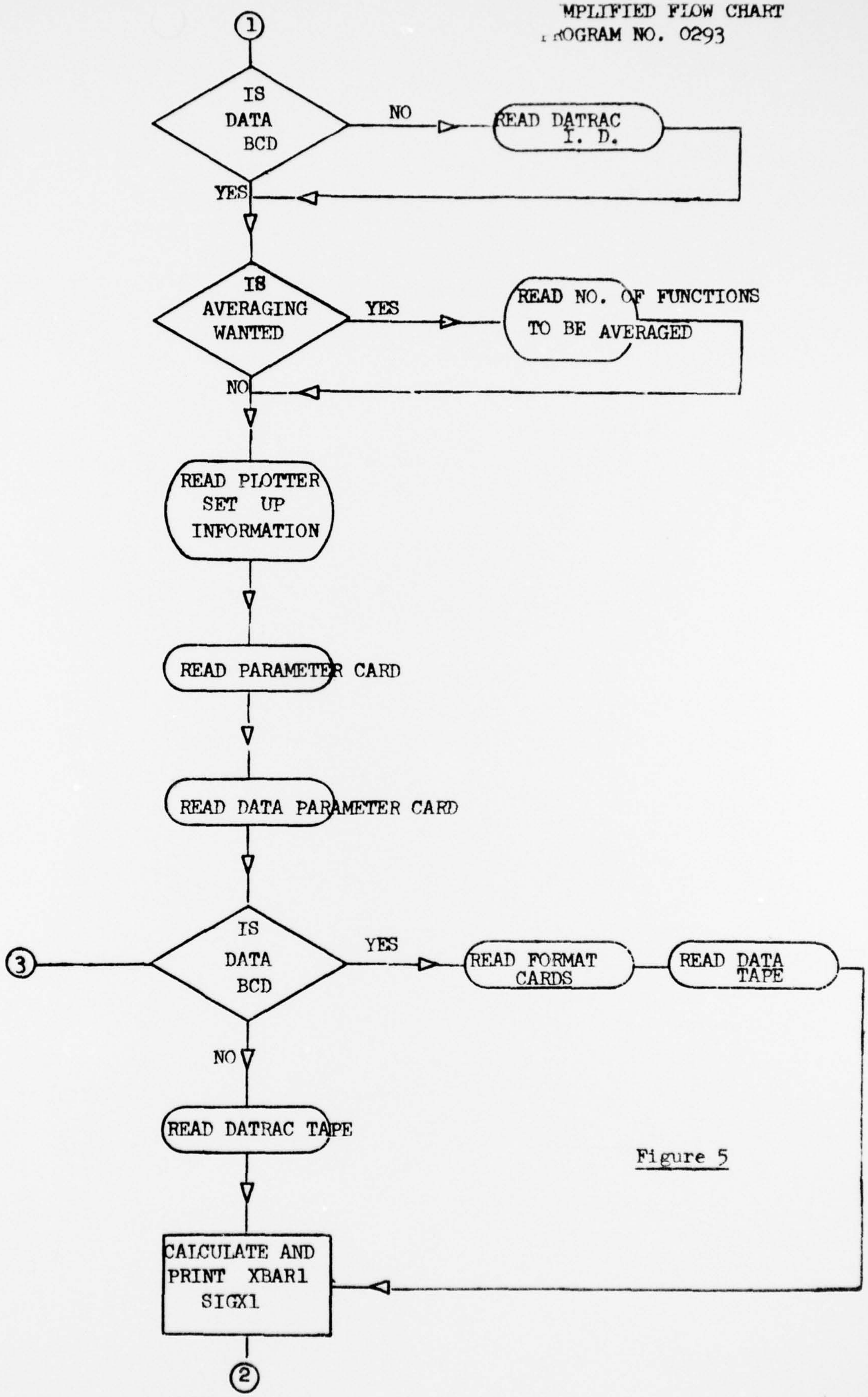
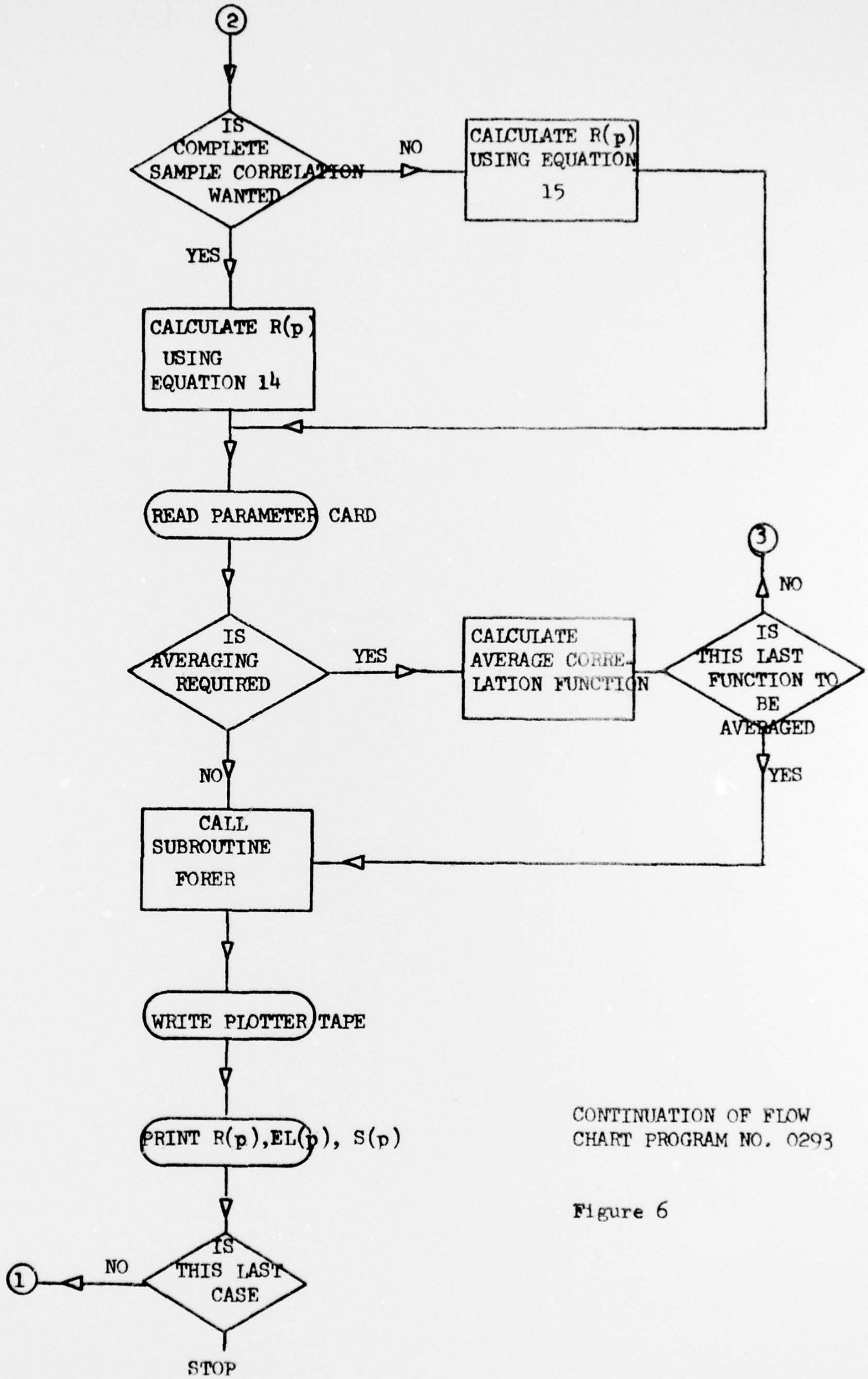

1
IS DATA BCD
NO
READ DATRAC I. D.
YES
IS AVERAGING WANTED
YES
READ NO. OF FUNCTIONS TO BE AVERAGED
NO
READ PLOTTER SET UP INFORMATION
READ PARAMETER CARD
READ DATA PARAMETER CARD
3
IS DATA BCD
YES
READ FORMAT CARDS
READ DATA TAPE
NO
READ DATRAC TAPE
CALCULATE AND PRINT XBAR1 SIGX1
2

Figure 5



CONTINUATION OF FLOW CHART PROGRAM NO. 0293

Figure 6

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